

QUANTIFYING MINIMAL NON-COLLINEARITY AMONG RANDOM POINTS

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ABSTRACT. Let $\varphi_{n,K}$ denote the largest angle in all the triangles with vertices among the n points selected at random in a compact convex subset K of \mathbb{R}^d with nonempty interior, where $d \geq 2$. It is shown that the distribution of the random variable $\lambda_d(K) \frac{n^3}{3!} (\pi - \varphi_{n,K})^{d-1}$, where $\lambda_d(K)$ is a certain positive real number which depends only on the dimension d and the shape of K , converges to the standard exponential distribution as $n \rightarrow \infty$. By using the Steiner symmetrization, it is also shown that $\lambda_d(K)$ – which is referred to in the paper as the elongation of K – attains its minimum if and only if K is a ball $B^{(d)}$ in \mathbb{R}^d . Finally, the asymptotics of $\lambda_d(B^{(d)})$ for large d is determined.

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1. Summary and discussion

Let K be a bounded convex subset of \mathbb{R}^d with nonempty interior, for some natural $d \geq 2$. Let P_1, P_2, \dots be random points drawn independently and uniformly from K . For each natural n , let T_n be the set of all subsets of the set $\{1, \dots, n\}$ of cardinality 3. For each $t = \{i, j, k\} \in T_n$, let X_t denote the largest angle (with possible angle values in the interval $[0, \pi]$) in the triangle $P_i P_j P_k$ with vertices P_i, P_j, P_k . Let

$$(1.1) \quad \lambda_d(K) := 3\rho_d \frac{\mathbb{E}|P_1 P_2|^d}{V_d(K)},$$

where

$$(1.2) \quad \rho_d := \frac{\pi^{d/2} \Gamma(d)}{2^{2d-1} \Gamma((d+1)/2) \Gamma(d+1/2)},$$

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$|P_1 P_2| := \|P_1 - P_2\|$ is the Euclidean distance between P_1 and P_2 , and V_d is the d -dimensional volume (that is, the Lebesgue measure on \mathbb{R}^d). Let

$$(1.3) \quad \varphi_{n,K} := \max_{t \in T_n} X_t,$$

the largest angle in all the triangles with vertices among the points P_1, \dots, P_n .

Theorem 1.1. *The random variable (r.v.)*

$$(1.4) \quad Y_{n,d} := \lambda_d(K) \frac{n^3}{3!} (\pi - \varphi_{n,K})^{d-1}$$

converges as $n \rightarrow \infty$ to an exponential r.v. in distribution:

$$P(Y_{n,d} > a) \rightarrow e^{-a}$$

for each real $a > 0$, and hence uniformly in real $a > 0$.

This theorem will be proved in Section 2.

In the particular case when $d = 2$ and K is a square in \mathbb{R}^2 , a version of Theorem 1.1 with an unspecified constant in place of $\lambda_d(K)$ was presented on the MathOverflow site [4].

Note that the constant $\lambda_d(K)$ is, naturally, invariant with respect to all translations, homotheties, and orthogonal transformations applied to the set K .

The conclusion of Theorem 1.1 can be rewritten as follows:

$$P(\varphi_{n,K} > c) - \left(1 - \exp \left\{ -\lambda_d(K)(\pi - c)^{d-1} \frac{n^3}{3!} \right\}\right) \xrightarrow{n \rightarrow \infty} 0$$

uniformly in $c \in (0, \pi)$. Since $1 - \exp\{-\lambda_d(K)(\pi - c)^{d-1} \frac{n^3}{3!}\}$ is increasing in $\lambda_d(K)$, we see that, the greater the constant $\lambda_d(K)$, the greater the largest angle $\varphi_{n,K}$ tends to be, in an asymptotic stochastic sense.

This provides a reason to refer to the constant $\lambda_d(K)$ as the *elongation (coefficient)* of the convex set K . In particular, if the convex set K is close to a straight line segment, then the volume $V_d(K)$ will be relatively small and hence the elongation $\lambda_d(K)$ will be large; accordingly, the largest angle $\varphi_{n,K}$ will then tend to be close to its maximum possible value, π . Another justification for using this term, elongation, is provided by

Proposition 1.2. *Take any natural d . The minimum of the elongation $\lambda_d(K)$ over all bounded convex subsets K of \mathbb{R}^d with nonempty interior is attained only when the closure of K is a closed ball in \mathbb{R}^d of a positive radius.*

Proposition 1.2 will be proved in Appendix A. The proof is based on the Steiner symmetrization; see e.g. [3]. In particular, we shall show that the Steiner symmetrization does not increase the elongation.

In view of the foregoing discussion, the following appears natural:

Conjecture. *The largest angle $\varphi_{n,K}$ becomes stochastically smaller after any Steiner symmetrization of K . That is, for any natural d , any natural $n \geq 3$, any convex subset K of \mathbb{R}^d , any unit vector u in \mathbb{R}^d , and any $c \in (0, \pi)$, one has*

$$P(\varphi_{n,S_u K} > c) \leq P(\varphi_{n,K} > c),$$

where S_u denotes the Steiner symmetrization along the vector u .

Of course, one may also define the relative elongation of K , by dividing the elongation of K by that of the unit ball. Then Proposition 1.2 can be restated as follows: the relative elongation of K is never less than 1, and it equals 1 if and only if the closure of K is a closed ball in \mathbb{R}^d of a positive radius.

The asymptotics (for $d \rightarrow \infty$) of the elongation coefficient of balls in \mathbb{R}^d is presented in

Proposition 1.3. *Let $B^{(d)}$ denote the unit ball in \mathbb{R}^d . Let P_1 and P_2 be random points drawn independently and uniformly from $B^{(d)}$. Then*

$$(1.5) \quad \mathbb{E} |P_1 P_2|^d \sim \frac{1}{\sqrt{6}} \left(\frac{8}{3\sqrt{3}} \right)^d \quad \text{and} \quad \lambda_d(B^{(d)}) \sim \sqrt{3} \left(\frac{2}{3\sqrt{3}} \right)^d \quad \text{as } d \rightarrow \infty.$$

This proposition will be proved in Appendix B.

In particular, it follows from Proposition 1.3 that the L_d norm $\|P_1 P_2\|_d := (\mathbb{E} |P_1 P_2|^d)^{1/d}$ of the distance $|P_1 P_2|$ between the random points P_1 and P_2 in the unit ball converges (as $d \rightarrow \infty$) to $\frac{8}{3\sqrt{3}} = 1.539\dots$, which is strictly less than the diameter 2 of the unit ball. Mainly, this is a consequence of the fact that the maximum of the function ψ in (B.7) occurs at the point $t_* = -\frac{1}{3}$.

Let Dlim denote the limit in distribution, for $\text{Dlim}_{n \rightarrow \infty} Y_n \stackrel{\text{D}}{=} Y$ to mean that $Y_n \xrightarrow[n \rightarrow \infty]{} Y$ in distribution, where Y and the Y_n 's are real-valued r.v.'s. Next,

let Dlimsup denote the limit superior in distribution, for $\text{Dlimsup}_{n \rightarrow \infty} Y_n \stackrel{\text{D}}{\leq} Y$ to mean that $\limsup_{n \rightarrow \infty} \mathbb{P}(Y_n \geq y) \leq \mathbb{P}(Y \geq y)$ for all real y . Similarly, one may let $\text{Dliminf}_{n \rightarrow \infty} Y_n \stackrel{\text{D}}{\geq} Y$ mean that $\text{Dlimsup}_{n \rightarrow \infty} (-Y_n) \stackrel{\text{D}}{\leq} -Y$ or, equivalently, $\liminf_{n \rightarrow \infty} \mathbb{P}(Y_n > y) \geq \mathbb{P}(Y > y)$ for all real y . Then $\text{Dlim}_{n \rightarrow \infty} Y_n \stackrel{\text{D}}{=} Y$ if and only if $\text{Dlimsup}_{n \rightarrow \infty} Y_n \stackrel{\text{D}}{\leq} Y$ and $\text{Dliminf}_{n \rightarrow \infty} Y_n \stackrel{\text{D}}{\geq} Y$. In the case when the r.v. Y is actually non-random, here one may simply write \leq , \geq , and $=$ instead of the symbols $\stackrel{\text{D}}{\leq}$, $\stackrel{\text{D}}{\geq}$, and $\stackrel{\text{D}}{=}$ (respectively). In these terms, one can now state the following immediate corollary of Theorem 1.1 and Propositions 1.2 and 1.3.

Corollary 1.4.

(1.6)

$$\text{Dlim}_{d \rightarrow \infty} \text{Dlim}_{n \rightarrow \infty} (n^{3/(d-1)} (\pi - \varphi_{n, B^{(d)}})) = \frac{3\sqrt{3}}{2} \geq \text{Dlimsup}_{d \rightarrow \infty} \text{Dlim}_{n \rightarrow \infty} (n^{3/(d-1)} (\pi - \varphi_{n, K_d})),$$

where, for each natural $d \geq 2$, K_d is an arbitrary bounded convex subset of \mathbb{R}^d with nonempty interior.

Note that the double-limit value/bound $\frac{3\sqrt{3}}{2}$ in (1.6) is non-random; therefore, the symbol D is not overset there. Somewhat crudely but perhaps more transparently, (1.6) may be expressed in terms of an approximate equality \approx and an “approximate inequality” \lesssim , as follows: for large d and very large n ,

$$\varphi_{n, B^{(d)}} \approx \pi - \frac{3\sqrt{3}/2}{n^{3/(d-1)}} \lesssim \varphi_{n, K_d},$$

with non-random approximate value/bound $\pi - \frac{3\sqrt{3}/2}{n^{3/(d-1)}}$.

2. Proof of Theorem 1.1

Without loss of generality (wlog), (i) the diameter of the set K (defined as $\sup\{\|x-y\|: x, y \text{ in } K\}$) equals 1 and (ii) the point $0 \in \mathbb{R}^d$ is in the interior of K , so that $B_0(r) \subseteq K$ for some real $r > 0$; here, as usual, $B_x(r) := \{y \in \mathbb{R}^d: \|x-y\| < r\}$ for any $x \in \mathbb{R}^d$.

Lemma 2.1. *Take any $\delta \in (0, 1)$ and any $x \in K_\delta$, where*

$$K_\delta := (1 - \delta)K := \{(1 - \delta)z: z \in K\}.$$

Then $B_x(r\delta) \subseteq K$.

Proof of Lemma 2.1. Take any $y \in B_x(r\delta)$. Then

$$y = x + (y - x) = (1 - \delta)\frac{x}{1 - \delta} + \delta\frac{y - x}{\delta} \in K,$$

because $\frac{x}{1 - \delta} \in K$, $\frac{y - x}{\delta} \in B_0(r) \subseteq K$, and the set K is convex. Thus, indeed $B_x(r\delta) \subseteq K$. \square

Take any

$$(2.1) \quad \varepsilon \in (0, \pi/2).$$

Take any distinct points x and y in \mathbb{R}^d , and let

$$(2.2) \quad \ell := \|x - y\|.$$

Let $S_{\varepsilon; x, y}$ denote the set of all points $z \in \mathbb{R}^d$ such that the angle A_{xzy} at the vertex z in the triangle xyz is $> \pi - \varepsilon$. In view of the condition $\varepsilon \in (0, \pi/2)$, one may note that for any $z \in S_{\varepsilon; x, y}$ the angle A_{xzy} coincides with the largest angle (denoted here by $A_{\{x, y, z\}}$) in the triangle xyz . Let

$$V_{\varepsilon; d; x, y} := V_d(S_{\varepsilon; x, y}).$$

Lemma 2.2. *One has*

$$V_{\varepsilon; d; x, y} \sim \rho_d \ell^d \varepsilon^{d-1},$$

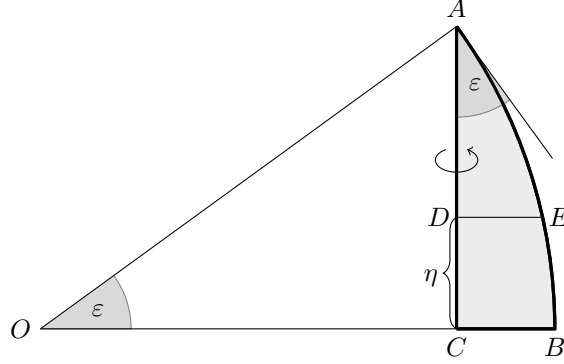
where ρ_d and ℓ are as in (1.2) and (2.2).

Here and in what follows, it is assumed by default that the limit relations are as $\varepsilon \downarrow 0$, unless specified otherwise. For any positive expressions \mathcal{E}_1 and \mathcal{E}_2 , we write $\mathcal{E}_1 \sim \mathcal{E}_2$ if $\mathcal{E}_1/\mathcal{E}_2 \rightarrow 1$, $\mathcal{E}_1 \lesssim \mathcal{E}_2$ if $\limsup(\mathcal{E}_1/\mathcal{E}_2) < \infty$, and $\mathcal{E}_1 \ll \mathcal{E}_2$ if $\mathcal{E}_1/\mathcal{E}_2 \rightarrow 0$.

Proof of Lemma 2.2. By homothety, wlog $\ell = 2$. Note that the angle A_{xzy} equals $\pi - \varepsilon$ if and only if the arc of the circle through the points x, z, y (in this order) is such that the smaller one of the two adjacent angles between the line xy (through the points x, y) and the line tangent to the mentioned arc at the point x equals ε (of course, the roles of x and y are interchangeable here). It follows that $V_{\varepsilon; d; x, y}$ equals twice the volume of the body of revolution obtained by the rotation in \mathbb{R}^d of the curvilinear triangle $AEBCDA$ about the line AC , where the coordinates of the points A, B, C, D, E in some orthonormal basis of some two-dimensional subspace of \mathbb{R}^d are as follows:

$$\begin{aligned} A &= (\cot \varepsilon, 1), & B &= (\csc \varepsilon, 0), & C &= (\cot \varepsilon, 0), \\ D &= (\cot \varepsilon, \eta), & E &= (\sqrt{\csc^2 \varepsilon - \eta^2}, \eta), \end{aligned}$$

$0 < \eta < 1$, the curve AEB is the arc of the circle centered at $O = (0, 0)$, and the curvilinear triangle $AEB CDA$ is the convex hull of the union of the arc AEB and the singleton set $\{C\}$. This is illustrated in the picture here.



Note that $|CD| = \eta$ and

$$|DE| = g(\eta) := \sqrt{\csc^2 \varepsilon - \eta^2} - \cot \varepsilon = \frac{1 - \eta^2}{\sqrt{\csc^2 \varepsilon - \eta^2} + \cot \varepsilon} \sim \frac{1}{2} (1 - \eta^2) \varepsilon.$$

So, letting β_{d-1} denote the volume of the unit ball in \mathbb{R}^{d-1} , we have

$$V_{\varepsilon; d; x, y} = 2 \int_0^1 \beta_{d-1} g(\eta)^{d-1} d\eta \sim 2 \frac{\beta_{d-1} \varepsilon^{d-1}}{2^{d-1}} \int_0^1 (1 - \eta^2)^{d-1} d\eta = \rho_d 2^d \varepsilon^{d-1},$$

because

$$(2.3) \quad \beta_{d-1} = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)} \quad \text{and} \quad \int_0^1 (1 - \eta^2)^{d-1} d\eta = \frac{\sqrt{\pi} \Gamma(d)}{2\Gamma(d+1/2)}.$$

Thus, Lemma 2.2 is proved. \square

Lemma 2.3. *For the conditional probability that $X_{\{1,2,3\}} > \pi - \varepsilon$ given P_1 and P_2 , one has*

$$(2.4) \quad \mathbb{P}(X_{\{1,2,3\}} > \pi - \varepsilon | P_1, P_2) \leq \varepsilon^{d-1}.$$

Proof of Lemma 2.3. The event $X_{\{1,2,3\}} > \pi - \varepsilon$ implies that

$$(2.5) \quad \text{the distance from } P_3 \text{ to the line } P_1 P_2 \text{ is } \leq (|P_1 P_3| \wedge |P_2 P_3|) \sin \varepsilon \leq \varepsilon,$$

since the diameter of the set K was assumed to equal 1; so, then the point P_3 lies in the intersection (say I) of K with (the convex hull of) the round cylinder of radius ε (with axis $P_1 P_2$). The intersection of I with any line parallel to $P_1 P_2$ is a segment of length not exceeding the diameter of K , which is 1. So, $V_d(I) \leq \varepsilon^{d-1}$, and hence (2.4) follows. \square

Lemma 2.4. *For any t and s in T_n such that $t \neq s$ and $t \cap s \neq \emptyset$, one has*

$$\mathbb{P}(X_t > \pi - \varepsilon, X_s > \pi - \varepsilon) \leq \varepsilon^{2(d-1)},$$

uniformly over all such s and t .

Proof of Lemma 2.4. Take any t and s in T_n such that $t \neq s$ and $t \cap s \neq \emptyset$. Then the cardinality of $t \cap s$ is either 2 or 1. So, wlog one of the following two cases holds: (i) $t = \{1, 2, 3\}$ and $s = \{1, 2, 4\}$ or (ii) $t = \{1, 2, 3\}$ and $s = \{1, 4, 5\}$. In case (i),

$$\begin{aligned} \mathbb{P}(X_t > \pi - \varepsilon, X_s > \pi - \varepsilon) &= \mathbb{E} \mathbb{I}\{X_{\{1,2,3\}} > \pi - \varepsilon\} \mathbb{P}(X_{\{1,2,4\}} > \pi - \varepsilon | P_1, P_2, P_3) \\ &= \mathbb{E} \mathbb{I}\{X_{\{1,2,3\}} > \pi - \varepsilon\} \mathbb{P}(X_{\{1,2,4\}} > \pi - \varepsilon | P_1, P_2) \\ &\leq \mathbb{E} \mathbb{I}\{X_{\{1,2,3\}} > \pi - \varepsilon\} \varepsilon^{d-1} \\ &= \mathbb{E} \mathbb{P}(X_{\{1,2,3\}} > \pi - \varepsilon | P_1, P_2) \varepsilon^{d-1} \leq \varepsilon^{2(d-1)}, \end{aligned}$$

by Lemma 2.3; here and elsewhere, $\mathbb{I}\{\cdot\}$ is the indicator function. Similarly, in case (ii),

$$\begin{aligned} \mathbb{P}(X_t > \pi - \varepsilon, X_s > \pi - \varepsilon) &= \mathbb{E} \mathbb{I}\{X_{\{1,2,3\}} > \pi - \varepsilon\} \mathbb{P}(X_{\{1,4,5\}} > \pi - \varepsilon | P_1, P_2, P_3) \\ &= \mathbb{E} \mathbb{I}\{X_{\{1,2,3\}} > \pi - \varepsilon\} \mathbb{P}(X_{\{1,4,5\}} > \pi - \varepsilon | P_1) \\ &= \mathbb{E} \mathbb{I}\{X_{\{1,2,3\}} > \pi - \varepsilon\} \mathbb{E}(\mathbb{P}(X_{\{1,4,5\}} > \pi - \varepsilon | P_1, P_4) | P_1) \\ &\leq \mathbb{E} \mathbb{I}\{X_{\{1,2,3\}} > \pi - \varepsilon\} \varepsilon^{d-1} \leq \varepsilon^{2(d-1)}. \end{aligned}$$

Lemma 2.4 is now proved. \square

Lemma 2.5. *One has*

$$(2.6) \quad \mathbb{P}(X_{\{1,2,3\}} > \pi - \varepsilon) \sim \lambda_d(K) \varepsilon^{d-1}.$$

Proof of Lemma 2.5. The event $\{X_{\{1,2,3\}} > \pi - \varepsilon\}$ is the union of the events $\{A_{P_1 P_3 P_2} > \pi - \varepsilon\}$, $\{A_{P_1 P_2 P_3} > \pi - \varepsilon\}$, and $\{A_{P_2 P_1 P_3} > \pi - \varepsilon\}$, and these events are pairwise disjoint, in view of (2.1). So,

$$(2.7) \quad \mathbb{P}(X_{\{1,2,3\}} > \pi - \varepsilon) = 3 \mathbb{P}(A_{P_1 P_3 P_2} > \pi - \varepsilon).$$

By Lemma 2.1 with

$$\delta := \varepsilon/r,$$

the event $\{\{P_1, P_2\} \subseteq K_\delta\}$ implies $B_{P_1}(\varepsilon) \cup B_{P_2}(\varepsilon) \subseteq K$. On the other hand, in view of (2.1), the event $\{A_{P_1 P_3 P_2} > \pi - \varepsilon\}$ implies that the point on the line $P_1 P_2$ that is the closest one to the P_3 is between P_1 and P_2 . Hence, in view (2.5), the event $\{A_{P_1 P_3 P_2} > \pi - \varepsilon, \{P_1, P_2\} \subseteq K_\delta\}$ implies that P_3 is in the convex hull of the set $B_{P_1}(\varepsilon) \cup B_{P_2}(\varepsilon)$ and thus in K . So, on the event $\{\{P_1, P_2\} \subseteq K_\delta\}$,

$$\mathbb{P}(A_{P_1 P_3 P_2} > \pi - \varepsilon | P_1, P_2) = \frac{V_{\varepsilon; d; P_1, P_2}}{V_d(K)} \sim \rho_d \frac{|P_1 P_2|^d}{V_d(K)} \varepsilon^{d-1}$$

by Lemma 2.2, whence

$$\begin{aligned} (2.8) \quad \mathbb{P}(A_{P_1 P_3 P_2} > \pi - \varepsilon, \{P_1, P_2\} \subseteq K_\delta) &\sim \rho_d \frac{\varepsilon^{d-1}}{V_d(K)} \mathbb{E} |P_1 P_2|^d \mathbb{I}\{\{P_1, P_2\} \subseteq K_\delta\} \\ &= \rho_d \frac{\varepsilon^{d-1}}{V_d(K)} (1 - \delta)^{3d} \mathbb{E} |P_1 P_2|^d \sim \rho_d \frac{\varepsilon^{d-1}}{V_d(K)} \mathbb{E} |P_1 P_2|^d, \end{aligned}$$

here are details concerning the scaling factor $(1 - \delta)^{3d}$ in the above display: for any $t \in (0, 1)$,

$$\begin{aligned} \mathbb{E} |P_1 P_2|^d \mathbb{I}\{\{P_1, P_2\} \subseteq tK\} &= \int_{tK} \int_{tK} \|x_1 - x_2\|^d \frac{V_d(dx_1)}{V_d(K)} \frac{V_d(dx_2)}{V_d(K)} \\ &= \int_K \int_K \|ty_1 - ty_2\|^d \frac{V_d(t dy_1)}{V_d(K)} \frac{V_d(t dy_2)}{V_d(K)} \\ &= t^{3d} \int_K \int_K \|y_1 - y_2\|^d \frac{V_d(dy_1)}{V_d(K)} \frac{V_d(dy_2)}{V_d(K)} = t^{3d} \mathbb{E} |P_1 P_2|^d. \end{aligned}$$

Note next that $\mathbb{P}(P_1 \notin K_\delta) = 1 - (1 - \delta)^d \ll 1$. On the other hand, by Lemma 2.3, $\mathbb{P}(A_{P_1 P_3 P_2} > \pi - \varepsilon | P_1) \leq \varepsilon^{d-1}$. So,

$$\begin{aligned} &\mathbb{P}(A_{P_1 P_3 P_2} > \pi - \varepsilon, P_2 \notin K_\delta) \\ (2.9) \quad &= \mathbb{P}(A_{P_1 P_3 P_2} > \pi - \varepsilon, P_1 \notin K_\delta) \leq \varepsilon^{d-1} \mathbb{P}(P_1 \notin K_\delta) \ll \varepsilon^{d-1}. \end{aligned}$$

Now Lemma 2.5 immediately follows by (2.7), (2.8), (2.9), and (1.1). \square

To complete the proof of Theorem 1.1, we shall use the key result of Galambos [2], which is in turn based on a combinatorial graph sieve theorem due to Rényi [5]. For readers' convenience, let us restate here the result of [2], as follows.

For each natural n , let T_n be a set of cardinality $|T_n| \rightarrow \infty$ and let E_n be a set of subsets of T_n of cardinality 2 such that

$$(2.10) \quad |E_n| \ll |T_n|^2.$$

In this setting, all limit relations are stated for $n \rightarrow \infty$.

For natural n and k and for $\alpha \in \{0, 1\}$, let $H_{n,k}^{(\alpha)}$ denote the set of all subsets F of T_n of cardinality $|F| = k$ such that exactly α subsets of F of cardinality 2 belong to E_n .

Take any real number $a > 0$, and let a sequence of real numbers $c_n = c_n(a)$ be such that the following conditions hold:

$$(2.11) \quad \sum_{t \in T_n} \mathbb{P}(X_t \geq c_n) \rightarrow a;$$

$$(2.12) \quad \sup_{n, t \in T_n} |T_n| \mathbb{P}(X_t \geq c_n) < \infty;$$

$$(2.13) \quad |E_n| \max_{\{t,s\} \in E_n} \mathbb{P}(X_t \geq c_n, X_s \geq c_n) \rightarrow 0;$$

for each natural k

$$(2.14) \quad \sum_{F \in H_{n,k}^{(0)}} \left[\mathbb{P}\left(\min_{t \in F} X_t \geq c_n\right) - \prod_{t \in F} \mathbb{P}(X_t \geq c_n) \right] \rightarrow 0;$$

and for each natural k there is a real number d_k such that for all $F \in H_{n,k}^{(1)}$

$$(2.15) \quad \mathbb{P}\left(\min_{t \in F} X_t \geq c_n\right) \leq d_k \mathbb{P}(X_s \geq c_n, X_r \geq c_n) \prod_{t \in F \setminus \{s,r\}} \mathbb{P}(X_t \geq c_n),$$

where $\{s, r\}$ is the only subset of F of cardinality 2 that belongs to E_n .

Then

$$(2.16) \quad \mathbb{P} \left(\max_{t \in T_n} X_t < c_n \right) \longrightarrow e^{-a}.$$

(The theorem in [2] was stated in terms of (sub)sequences rather than subsets, but the formulation given above is easily seen to be equivalent to that in [2].)

The four conditions (2.10), (2.13), (2.14), and (2.15) specify a notion of weak dependence of the r.v.'s X_t . In particular, in the case when the X_t 's are independent, it is easy to see that none of these four conditions is needed to deduce (2.16) already from (2.11) and (2.12).

Now we are ready to complete the proof of Theorem 1.1. Indeed, let T_n and X_t be as described in the beginning of Section 1, so that

$$(2.17) \quad |T_n| = \binom{n}{3} \sim \frac{n^3}{3!}$$

and the r.v.'s X_t are exchangeable. Next, let E_n be the set of subsets $\{t, s\}$ of T_n of cardinality 2 such that $t \cap s \neq \emptyset$.

Then $|E_n| \leq n^3 \cdot n^2$, so that condition (2.10) holds.

Take now indeed any real $a > 0$ and let

$$(2.18) \quad c_n := c_n(a) := \pi - \varepsilon_n, \quad \text{where} \quad \varepsilon_n := \left(\frac{a}{|T_n| \lambda_d(K)} \right)^{1/(d-1)},$$

so that $c_n \uparrow \pi$; from now on, all limit relations are stated for $n \rightarrow \infty$. Then conditions (2.11) and (2.12) hold by Lemma 2.5. Condition (2.13) follows by Lemma 2.4 and (2.10). Condition (2.14) is trivial here, because for each $F \in H_{n,k}^{(0)}$ the family of r.v.'s $(X_t)_{t \in F}$ is independent. Finally, condition (2.15) holds (with $d_k = 1$) because for each $F \in H_{n,k}^{(1)}$ and s, r as described in that condition, the family of r.v.'s $(X_t)_{t \in F \setminus \{s, r\}}$ is independent in itself and also independent of the random pair (X_s, X_r) .

Thus, the conclusion (2.16) holds, with c_n as in (2.18). In view of (1.4) and (1.3),

$$\mathbb{P} \left(\max_{t \in T_n} X_t < c_n \right) = \mathbb{P} \left(Y_{n,d} \frac{|T_n|}{n^3/3!} > a \right).$$

So, to finish the proof of Theorem 1.1, it remains to recall (2.17). \square

Appendix A. Proof of Proposition 1.2

As mentioned before, this proof is based on the Steiner symmetrization.

Let $E_d(K)$ denote $\mathbb{E} |P_1 P_2|^d$, where, as before, P_1 and P_2 are random points drawn independently and uniformly from K . Let \overline{K} denote the closure of K . Then for any real $\delta > 0$ one has $K \subseteq \overline{K} \subseteq (1 + \delta)K$, whence

$$E_d(K) \leq E_d(\overline{K}) \leq E_d((1 + \delta)K) = (1 + \delta)^d E_d(K) \xrightarrow[\delta \downarrow 0]{} E_d(K),$$

so that $E_d(\overline{K}) = E_d(K)$. Similarly, $V_d(\overline{K}) = V_d(K)$. So, wlog the bounded convex set K is closed and hence compact, which will be assumed henceforth.

The Steiner symmetrization can be described as follows. Take any unit vector $u \in \mathbb{R}^d$. For each $x \in \mathbb{R}^d$, there are uniquely determined $\theta(x) = \theta_u(x)$ in \mathbb{R} and

$H(x) = H_u(x)$ in the orthogonal complement $\{u\}^\perp := \{y \in \mathbb{R}^d : \langle y, u \rangle = 0\}$ of the singleton set $\{u\}$ to \mathbb{R} such that

$$(A.1) \quad x = \theta(x)u + H(x).$$

For each $y \in \{u\}^\perp$, let

$$s(y) := s_{K,u}(y) := \{\alpha \in \mathbb{R} : \alpha u + y \in K\}.$$

Let then

$$Y_u := Y_{K,u} := \{y \in \{u\}^\perp : s(y) \neq \emptyset\}.$$

Clearly, for each $y \in \{u\}^\perp$, the set $s(y)$ is a compact convex subset of \mathbb{R} , and this set is nonempty if $y \in \mathcal{Y}_u$. So, for each $y \in \mathcal{Y}_u$ there exist unique real numbers $a(y) = a_K(y)$ and $b(y) = b_K(y)$ such that $a(y) \leq b(y)$ and

$$s(y) = [a(y), b(y)].$$

Then the Steiner symmetrization, say $S_u K$, of K along the unit vector u is the set

$$S_u K := \bigcup_{y \in Y_{K,u}} \left\{ \alpha u + y : \frac{a(y) - b(y)}{2} \leq \alpha \leq \frac{b(y) - a(y)}{2} \right\}.$$

It is well known and easy to see that $S_u K$ is a compact convex set, of the same volume as K :

$$V_d(S_u K) = V_d(K).$$

One may also note that the compact convex set $S_u K$ is determined by the conditions

$$Y_{S_u K, u} = Y_{K, u} \quad \text{and} \quad s_{S_u K, u}(y) = \left[\frac{a(y) - b(y)}{2}, \frac{b(y) - a(y)}{2} \right]$$

for all $y \in Y_{K, u}$.

Now we are ready to state

Lemma A.1. $E_d(S_u K) \leq E_d(K)$. Moreover, $E_d(S_u K) = E_d(K)$ if and only if K is a ball.

Proof of Lemma A.1. For the random points P_1 and P_2 as before and $j \in \{1, 2\}$, in view of (A.1) one can write

$$P_j = \xi_j u + Q_j, \quad \text{where } \xi_j := \theta(P_j) \text{ and } Q_j := H(P_j).$$

Clearly, the random pairs (ξ_1, Q_1) and (ξ_2, Q_2) are independent copies of each other. Also, conditionally on (Q_1, Q_2) , the r.v.'s ξ_1 and ξ_2 are independent, and for each $j \in \{1, 2\}$ the conditional distribution of ξ_j given (Q_1, Q_2) is uniform over the interval $[A_j, B_j] := [a(Q_j), b(Q_j)]$.

Let now $\tilde{P}_j, \tilde{\xi}_j, \tilde{Q}_j, \tilde{A}_j, \tilde{B}_j$ be defined similarly to $P_j, \xi_j, Q_j, A_j, B_j$ (respectively), but with $S_u K$ in place of K . Note that the random pairs (Q_1, Q_2) and $(\tilde{Q}_1, \tilde{Q}_2)$ are the same in distribution, and we may and will assume that they are just equal to each other: $(\tilde{Q}_1, \tilde{Q}_2) = (Q_1, Q_2)$. Then, given (Q_1, Q_2) , for each $j \in \{1, 2\}$ the conditional distribution of ξ_j is the same as that of $\tilde{\xi}_j + \frac{A_j + B_j}{2}$. So, letting

$$\eta := \xi_1 - \xi_2, \quad \tilde{\eta} := \tilde{\xi}_1 - \tilde{\xi}_2, \quad C := \frac{A_1 + B_1}{2} - \frac{A_2 + B_2}{2},$$

we see that, given (Q_1, Q_2) , the conditional distribution of η is the same as that of $\tilde{\eta} + C$, whereas the conditional distribution of the real valued r.v. $\tilde{\eta}$ given (Q_1, Q_2)

is symmetric (about 0). So, introducing $Z := |Q_1 Q_2| = \|Q_1 - Q_2\|$ and writing the instance $|P_1 P_2|^2 = \eta^2 + Z^2$ of the Pythagoras theorem, we have

$$(A.2) \quad E_d(K) = \mathbb{E} g_{\tilde{\eta}, Z}(|C|) \quad \text{and} \quad E_d(S_u K) = \mathbb{E} g_{\tilde{\eta}, Z}(0),$$

where

$$g_{t,z}(c) := g_{t,z,d}(c) := \frac{1}{2}((t+c)^2 + z^2)^{d/2} + \frac{1}{2}((-t+c)^2 + z^2)^{d/2}.$$

for real t, z, c . For each $(t, z, d) \in \mathbb{R} \times (0, \infty) \times [1, \infty)$, the function $g_{t,z} = g_{t,z,d}$ is even and strictly convex on \mathbb{R} , and hence strictly increasing on $[0, \infty)$. In view of (A.2), this immediately yields the inequality $E_d(S_u K) \leq E_d(K)$.

Moreover, it follows that the equality $E_d(S_u K) = E_d(K)$ is possible only if $C = 0$ almost surely. By the Fubini theorem, this implies that there is some $y_1 \in Y_u$ such that for almost all $y \in Y_u$ one has

$$(A.3) \quad \frac{a(y) + b(y)}{2} = \tau_u := \frac{a(y_1) + b(y_1)}{2};$$

the dependence of the functions a and b on the unit vector u (and on the set K) is implicit here.

Let us now show that (A.3) holds for all $y \in Y_u$. Toward this end, note first that the set Y_u is the orthogonal projection of K onto $\{u\}^\perp$, and so, Y_u is convex and compact. Moreover, for any y_0 and y_1 in Y_u and any $t \in (0, 1)$ one has $b(y_j)u + y_j \in K$ for $j \in \{1, 2\}$, whence

$$[(1-t)b(y_0) + tb(y_1)]u + (1-t)y_0 + ty_1 = (1-t)(b(y_1)u + y_1) + t(b(y_2)u + y_2) \in K,$$

which yields $(1-t)b(y_0) + tb(y_1) \in s((1-t)y_0 + ty_1)$, so that $(1-t)b(y_0) + tb(y_1) \leq b((1-t)y_0 + ty_1)$. This shows that the function b is concave, on Y_u , and hence continuous on the interior $\text{int } Y_u$ of Y_u , by a well-known theorem (see e.g. [6, Theorem 10.1]). It follows that (A.3) holds for all $y \in \text{int } Y_u$.

Next, take any $y \in Y_u$. Take then $x := b(y)u + y$, so that $x \in K$. By Lemma 2.1, for all $\delta \in (0, 1)$ one has $\text{int } K \ni (1-\delta)x = (1-\delta)b(y)u + (1-\delta)y$, so that $(1-\delta)y$ is in the orthogonal projection of $\text{int } K$ onto $\{u\}^\perp$ and hence in $\text{int } Y_u$. It follows that

$$\frac{a((1-\delta)y) + b((1-\delta)y)}{2} = \tau_u,$$

for all $\delta \in (0, 1)$. By the compactness of K , there are real a_* and b_* and a sequence (δ_m) in $(0, 1)$ converging to 0 such that $b((1-\delta_m)y) \rightarrow b_*$ and $a((1-\delta_m)y) \rightarrow a_*$, whence $K \ni b((1-\delta_m)y)u + (1-\delta_m)y \rightarrow b_*u + y$. Therefore and because K is closed, we have $b_*u + y \in K$, and so, $b_* \leq b(y)$. On the other hand, by the concavity of b , $b((1-\delta_m)y) \geq (1-\delta_m)b(y) + \delta_m b(0) \rightarrow b(y)$, so that $b_* = \lim_m b((1-\delta_m)y) \geq b(y)$. Thus, $b(y) = b_* = \lim_m b((1-\delta_m)y)$ and, similarly, $a(y) = \lim_m a((1-\delta_m)y)$. Since $(1-\delta_m)y \in \text{int } Y_u$ and (A.3) was established for all $y \in \text{int } Y_u$, it follows that indeed (A.3) holds for any $y \in Y_u$.

This means that the set K is symmetric about the plane

$$\Pi_u := \{x \in \mathbb{R}^d : \langle x, u \rangle = \tau_u\},$$

for any unit vector $u \in \mathbb{R}^d$. By translation, $\text{wlog } \bigcap_{i=1}^d \Pi_{e_i} = \{0\}$, where (e_1, \dots, e_d) is (say) the standard basis of \mathbb{R}^d . Since K is symmetric about each of the ‘‘coordinate’’ hyperplanes $\Pi_{e_1}, \dots, \Pi_{e_d}$, it is easy to see that K is centrally symmetric about the origin.

So, taking any unit vector $v \in \mathbb{R}^d$, one has $a(v) + b(v) = 0$. It follows that the hyperplane Π_v of symmetry of K must pass through the origin. Thus, K is invariant with respect to the reflection in any hyperplane through the origin. On the other hand, by the Cartan–Dieudonné theorem (see e.g. [1]), any orthogonal transformation is the composition of reflections. We conclude that the compact convex set K is invariant with respect to any orthogonal transformation; hence, K is a ball.

To complete the proof of Lemma A.1, it remains to note that, if K is a ball, then obviously $E_d(\mathbf{S}_u K) = E_d(K)$. \square

Lemma A.2. *Let B denote the unit ball in \mathbb{R}^d , and suppose that $\mathbf{V}_d(K) = \mathbf{V}_d(B)$. Then $E_d(K) \geq E_d(B)$.*

Proof of Lemma A.2. Take any $\delta \in (0, 1)$. By a well-known result (see cf. [3, Theorem 1.5]), there exist a natural m and unit vectors u_1, \dots, u_m in \mathbb{R}^d such that

$$(1 + \delta)B \supseteq K_m := \mathbf{S}_{u_m} \cdots \mathbf{S}_{u_1} K \supseteq (1 - \delta)B.$$

So, in view of Lemma A.1,

$$\begin{aligned} E_d(K) &\geq E_d(K_m) \geq E_d((1 - \delta)B) \frac{\mathbf{V}_d((1 - \delta)B)}{\mathbf{V}_d(K_m)} \\ &\geq E_d((1 - \delta)B) \frac{\mathbf{V}_d((1 - \delta)B)}{\mathbf{V}_d((1 + \delta)B)} = \frac{(1 - \delta)^{2d}}{(1 + \delta)^d} E_d(B), \end{aligned}$$

for any $\delta \in (0, 1)$. Letting now $\delta \downarrow 0$, one immediately obtains Lemma A.2. \square

Now it is easy to finish the proof of Proposition 1.2. Indeed, by the scaling properties of $E_d(K) = \mathbf{E} |P_1 P_2|^d$ and $\mathbf{V}_d(K)$, wlog $\mathbf{V}_d(K) = \mathbf{V}_d(B)$, where B is the unit ball in \mathbb{R}^d , as in Lemma A.2. From that lemma and the definition (1.1) of $\lambda_d(K)$, it follows immediately that $\lambda_d(K) \geq \lambda_d(B)$.

Suppose now that $\lambda_d(K) = \lambda_d(B)$. Then $E_d(K) = E_d(B)$. On the other hand, $E_d(K) \geq E_d(\mathbf{S}_u K)$ by Lemma A.1, and $E_d(\mathbf{S}_u K) \geq E_d(B)$ by Lemma A.2, since $\mathbf{V}_d(\mathbf{S}_u K) = \mathbf{V}_d(K) = \mathbf{V}_d(B)$. Thus, $E_d(K) \geq E_d(\mathbf{S}_u K) \geq E_d(B) = E_d(K)$, whence $E_d(\mathbf{S}_u K) = E_d(K)$, so that, again by Lemma A.1, K is a ball. This completes the proof of Proposition 1.2.

Appendix B. Proof of Proposition 1.3

The random pair (P_1, P_2) equals $(R_1 U_1, R_2 U_2)$ in distribution, where R_1 and R_2 are real-valued r.v.'s each with density $\mathbb{R} \ni r \mapsto d r^{d-1} \mathbf{I}\{0 < r < 1\}$, U_1 and U_2 are random vectors each uniformly distributed on the unit sphere, and R_1, U_1, R_2, U_2 are independent. So, letting T denote the cosine of the angle between the random vectors U_1 and U_2 , we see that the r.v.'s R_1, R_2, T are independent, and $|P_1 P_2|^2$ equals $R_1^2 + R_2^2 - 2R_1 R_2 T$ in distribution. Moreover – because, by the spherical symmetry, T is independent of (say) U_1 – the density of T is $\mathbb{R} \ni t \mapsto C_d (1 - t^2)^{(d-3)/2} \mathbf{I}\{-1 < t < 1\}$, where

$$(B.1) \quad C_d := \frac{\Gamma(d/2)}{\sqrt{\pi} \Gamma((d-1)/2)} \sim \sqrt{\frac{d}{2\pi}};$$

all the limit relations in this proof are of course for $d \rightarrow \infty$. Hence,

$$(B.2) \quad \mathbf{E} |P_1 P_2|^d = C_d d^2 (J_{d,1} + J_{d,2}),$$

where

$$J_{d,1} := \int_{-1}^0 \Psi_d(t)(1-t^2)^{(d-3)/2} dt, \quad J_{d,2} := \int_0^1 \Psi_d(t)(1-t^2)^{(d-3)/2} dt,$$

$$\Psi_d(t) := \int_0^1 dr_1 \int_0^1 dr_2 r_1^{d-1} r_2^{d-1} (r_1^2 + r_2^2 - 2r_1 r_2 t)^{d/2}.$$

For $t > 0$, the integrand in the integral expression for $\Psi_d(t)$ is no greater than $2^{d/2}$, and so, $\Psi_d(t) \leq 2^{d/2}$. Therefore, for $d \geq 3$

$$(B.3) \quad J_{d,2} \leq 2^{d/2}.$$

Let us now estimate $J_{d,1}$. Note that

$$(B.4) \quad J_{d,1,1} \leq J_{d,1} \leq J_{d,1,1} + 2J_{d,1,2},$$

where

$$J_{d,1,j} := \int_{-1}^0 \Psi_{d,j}(t)(1-t^2)^{(d-3)/2} dt,$$

$$\Psi_{d,1}(t) := \int_{1-\delta}^1 dr_1 \int_{1-\delta}^1 dr_2 r_1^{d-1} r_2^{d-1} (r_1^2 + r_2^2 - 2r_1 r_2 t)^{d/2},$$

$$\Psi_{d,2}(t) := \int_0^{1-\delta} dr_1 \int_0^1 dr_2 r_1^{d-1} r_2^{d-1} (r_1^2 + r_2^2 - 2r_1 r_2 t)^{d/2},$$

$$\delta := \delta_d := 1/\sqrt{d-1}.$$

For $(r_1, r_2, t) \in (0, 1-\delta) \times (0, 1) \times (-1, 0)$, the integrand $r_1^{d-1} r_2^{d-1} (r_1^2 + r_2^2 - 2r_1 r_2 t)^{d/2}$ in $\Psi_{d,2}(t)$ does not exceed

$$(1-\delta)^{d-1} ((1-\delta)^2 + 1 - 2(1-\delta)t)^{d/2}$$

$$\leq (1-\delta)^{d-1} (1-t)^{d/2} 2^{d/2} \leq 2^{d/2} e^{-\sqrt{d-1}} (1-t)^{d/2},$$

and so, the latter expression is also an upper bound on $\Psi_{d,2}(t)$. It follows that

$$(B.5) \quad J_{d,1,2} \leq 2^{d/2} e^{-\sqrt{d-1}} Q_d,$$

where

$$Q_d := \int_{-1}^0 (1-t)^{d/2} (1-t^2)^{(d-3)/2} dt.$$

Letting $v := 1 - r_1$ and $w := 1 - r_2$, we have

$$\begin{aligned} r_1^2 r_2^2 (r_1^2 + r_2^2 - 2r_1 r_2 t) &= 2(1-t) - 6(1-t)(v+w) + O(v^2 + w^2) \\ &= 2(1-t)[1 - 3(v+w) + O(v^2 + w^2)] \\ &= 2(1-t) \exp\{-3(v+w) + O(v^2 + w^2)\} \\ &= 2(1-t) \exp\{-(3+o(1))(v+w)\} \end{aligned}$$

for $(r_1, r_2, t) \in (1-\delta, 1) \times (1-\delta, 1) \times (-1, 0)$, whence

$$\begin{aligned} \Psi_{d,1}(t) &\sim 2^{(d-1)/2} (1-t)^{(d-1)/2} (1^2 + 1^2 - 2 \times 1 \times 1 \times t)^{1/2} \\ &\quad \times \left(\int_0^\delta dv \exp\{-(3+o(1))v(d-1)/2\} \right)^2 \\ &\sim 2^{d/2} (1-t)^{d/2} \frac{4}{9d^2} \end{aligned}$$

and

$$J_{d,1,1} \sim 2^{d/2} \frac{4}{9d^2} Q_d.$$

Comparing this with (B.5), we see that $J_{d,1,2} \ll J_{d,1,1}$. Recalling now (B.4), we conclude that

$$(B.6) \quad J_{d,1} \sim J_{d,1,1} \sim 2^{d/2} \frac{4}{9d^2} Q_d.$$

The needed estimation of Q_d is straightforward. Indeed,

$$(B.7) \quad Q_d = \int_{-1}^0 (1-t)^{3/2} \exp\left\{\frac{d-3}{2} \psi(t)\right\} dt,$$

where $\psi(t) := 2\ln(1-t) + \ln(1+t)$. One has $\psi'(t) = -\frac{1+3t}{1-t^2}$; $\psi''(t) = -\frac{1}{(1+t)^2} - \frac{2}{(1-t)^2} < 0$; $\psi'(t) = 0 \iff t = t_* := -\frac{1}{3}$; $\exp\{\psi(t_*)\} = \frac{32}{27}$; and $\psi''(t_*) = -\frac{27}{8}$. So, standard reasoning yields

$$\begin{aligned} Q_d &\sim \int_{-1}^0 (1-t_*)^{3/2} \exp\left\{\frac{d-3}{2} \left[\psi(t_*) + \frac{1}{2}\psi''(t_*)(t-t_*)^2\right]\right\} dt \\ &\sim (1-t_*)^{3/2} \exp\left\{\frac{d-3}{2} \psi(t_*)\right\} \frac{\sqrt{2\pi}}{\sqrt{-\psi''(t_*)d/2}} = \sqrt{\pi} \frac{3\sqrt{3}}{4\sqrt{d}} \left(\frac{32}{27}\right)^{d/2}. \end{aligned}$$

Hence, by (B.6),

$$J_{d,1} \sim 2^{d/2} \frac{4}{9d^2} \sqrt{\pi} \frac{3\sqrt{3}}{4\sqrt{d}} \left(\frac{32}{27}\right)^{d/2} = \frac{\sqrt{\pi}}{d^2 \sqrt{3d}} \left(\frac{64}{27}\right)^{d/2}.$$

Comparing this with (B.3), we see that $J_{d,2} \ll J_{d,1}$. Now (B.2) and (B.1) yield

$$\mathbb{E}|P_1 P_2|^d \sim C_d d^2 J_{d,1} \sim C_d \frac{\sqrt{\pi}}{\sqrt{3d}} \left(\frac{64}{27}\right)^{d/2} \sim \frac{1}{\sqrt{6}} \left(\frac{8}{3\sqrt{3}}\right)^d,$$

which proves the first asymptotic equivalence in (1.5). The second asymptotic equivalence there now follows immediately by (1.1), (1.2), the first equality in (2.3) (with $d-1$ replaced by d), and the asymptotic equivalence $\Gamma(\alpha+1/2) \sim \sqrt{\alpha} \Gamma(\alpha)$ as $\alpha \rightarrow \infty$. \square

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